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QUOTIENT POLYTOPES OF CYCLIC POLYTOPES PART II: STABILITY OF THE f-VECTOR AND OF THE k-SKELETON

BY

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ABSTRACT

This paper is the continuation of an earlier paper on quotient polytopes $C(v,2m)/F$ of cyclic polytopes and the associated quotient complexes $\mathscr{C}(V,2m)/J$. Here, we study mainly what changes in the face J do not affect the f-vector of the quotient $\mathscr{C}(V,2m)/J$. In the last section we examine the corresponding question for k-skeleta, i.e., what changes in J do not affect the isomorphism type of $\mathrm{skel}_k\,\mathscr{C}(V,2m)/J$.

7. Introduction

In part I of this work we started to investigate the quotient polytopes $C(v,2m)/F$ of cyclic polytopes and the associated quotient complexes $\mathscr{C}(V,2m)/J$.

In this part we shall be dealing mainly with the following question: what changes in the face J do not affect the f-vector of the quotient $\mathscr{C}(V,2m)/J$? In the last section of this part (Section 12) we shall consider the corresponding question for k-skeleta, namely: what changes in J do not affect the isomorphism type of $\mathrm{skel}_k\,\mathscr{C}(V,2m)/J$?

Let us first recall the definition of $\mathscr{C}(V,2m)/J$ and its f-vector, and the notational convention associated with it. We assume that $V=\{1,\cdots,v\}$, $v>2m\geqq 2$, and consider the circuit $C(V)$, with edges $\{1,2\},\{2,3\},\cdots,\{v-1,v\},\{v,1\}$. $\mathscr{C}(V,2m)$ consists of all the subsets of those $2m$-subsets of V which contain only even blocks (see [1, Definition 3.1]). J is a *separated* subset of V, $0\leqq|J|=j\leqq m$, and

$$\mathscr{C}(V,2m)/J=\{S\subset V\setminus J : S\cup J\in\mathscr{C}(V,2m)\}.$$

Let $\mathscr{K}=\mathscr{C}(V,2m)/J$. Then $f(\mathscr{K})=(f_0(\mathscr{K}),\cdots,f_{2m-j-1}(\mathscr{K}))$, where

$$f_i(\mathscr{K})=|\{F\in\mathscr{K} : |F|=i+1\}|.$$

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The next definition is needed for the formulation of the main result of this part.

DEFINITION 7.1. Let J be a fixed subset of V.

(a) Two subsets A, B of V are *similar* if there is an automorphism of $C(V)$ (rotation or reflection) which maps A onto B.

(b) For a natural number α, and $x, y \in J$, let us write $x \leftrightarrow_\alpha y$ if the distance between x and y in $C(V)$ is $\leqq \alpha$, i.e., if $|x - y| \leqq \alpha$ or $|x - y| \geqq v - \alpha$.

(c) Let $\sim_\alpha$ be the transitive closure of $\leftrightarrow_\alpha$ on J, i.e., $x \sim_\alpha y$ iff there is a sequence $x = x_0, x_1, \cdots, x_t = y$ of elements of J ($t \geqq 0$), such that $x_{i-1} \leftrightarrow_\alpha x_i$ for $1 \leqq i \leqq t$.

(d) The equivalence classes of J with respect to $\sim_\alpha$ are called *α-blocks*.

(e) Two subsets J_1, J_2 of V are *α-equivalent* if there is 1-1 correspondence between the α-blocks of J_1 and those of J_2, so that corresponding α-blocks of J_1 and J_2 are similar in the sense of paragraph (a) above.

Figure 2 depicts a pair of 3-equivalent separated 8-subsets of V, each consisting of four 3-blocks. ($|V| = 26$.)

Now we are ready to state the main result of this part.

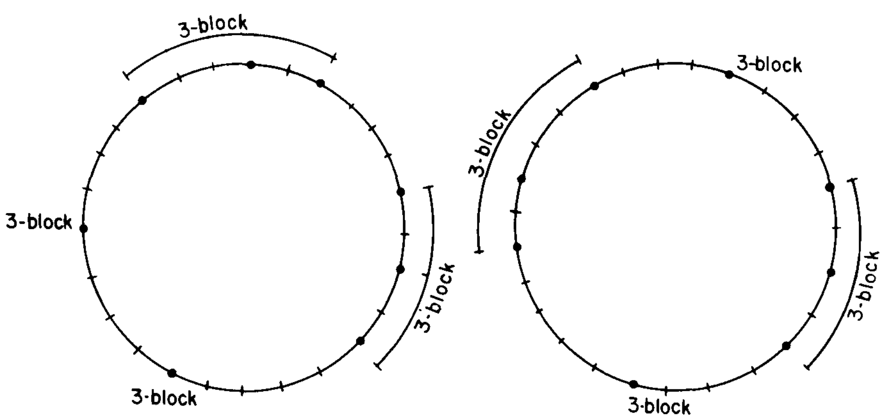


Fig. 2.

THEOREM 7.2. *Suppose $|V| = v > 2m \geqq 2$ and $0 \leqq j \leqq m$. If J and J' are separated j-subsets of V, and J, J' are $(m - j + 2)$-equivalent, then $f(\mathscr{C}(V, 2m)/J) = f(\mathscr{C}(V, 2m)/J')$.*

The proof of Theorem 7.2 falls into three parts. First, in Section 8, we deal with the "degenerate" cases where $j = m$ or $v \leqq 2m + 2$. In these cases the quotient $\mathscr{C}(V, 2m)/J$ is (isomorphic to the boundary complex of) a direct sum of simplices, and the $(m-j+2)$-equivalence of J and J' implies not only the equality of the f-vectors, but also the isomorphism of the quotients $\mathscr{C}(V,2m)/J$ and $\mathscr{C}(V,2m)/J'$.

Section 9 contains two lemmas which are used in the second part of the proof. In the second part (Section 10) we prove Theorem 7.2 in the case where J' is obtained from J by shifting the $(m-j+2)$-blocks of J along $C(V)$, keeping their relative order fixed.

Finally, in Section 11, we prove Theorem 7.2 in its full generality.

8. The cases $v<2m+3$ or $j=m$

THEOREM 8.1. *For $v<2m+3$, the hypotheses of Theorem 7.2 imply the isomorphism of the quotients $\mathscr{C}(V,2m)/J$ and $\mathscr{C}(V,2m)/J'$.*

PROOF. If $v=2m+1$, then $\mathscr{C}(V,2m)$ is (isomorphic to the boundary complex of) a $2m$-simplex, and the quotients $\mathscr{C}(V,2m)/J$, $\mathscr{C}(V,2m)/J'$ are both $(2m-j)$-simplices.

Consider the case $v=2m+2$. Since J and J' are $(m-j+2)$-equivalent, they have the same number, say γ, of $(m-j+2)$-blocks. If $\gamma=1$, then J and J' are similar subsets of V, and the corresponding quotients are clearly isomorphic. If $\gamma\geqq 2$, then $V\setminus J$ contains at least $\gamma(m-j+2)$ points between the different blocks, and at least $j-\gamma$ points "within" the blocks. (Obviously $\gamma\leqq j$. Since J is separated, a block which contains x points of J bounds at least $x-1$ separating points of $V\setminus J$.) Thus

$$2m+2-j=|V\setminus J|\geqq\gamma(m-j+2)+j-\gamma \tag{8.1.1}$$

and equality holds only if every two adjacent $(m-j+2)$-blocks of J are separated by precisely $m-j+2$ points of $V\setminus J$, and any two adjacent points within a block of J are separated by a single point of $V\setminus J$.

The inequality (8.1.1) is equivalent to $\gamma\leqq 2$. Since $\gamma\geqq 2$, we conclude that $\gamma=2$, and equality must hold in (8.1.1).

Therefore J consists of two $(m-j+1)$-blocks J_1 and J_2, separated by precisely $m-j+2$ points of $V\setminus J$ on each side, and two adjacent points within each block are separated by a single point of $V\setminus J$. The same holds for the two $(m-j+2)$-blocks J'_1 and J'_2 of J'.

Since, by assumption, J and J' have similar blocks, it follows that $|J_1|=|J'_1|$ and $|J_2|=|J'_2|$ (or $|J_1|=|J'_2|$ and $|J_2|=|J'_1|$), and therefore J and J' are similar subsets of V. This implies the isomorphism of the corresponding quotients. □

In the next theorem we settle the case $j=m$.

THEOREM 8.2. *Suppose* $|V| = v > 2m \geqq 2$, *and let* J, J' *be separated* m-*subsets of* V. *Let* $\mathcal{K} = \mathcal{C}(V, 2m)/J$, $\mathcal{K}' = \mathcal{C}(V, 2m)/J'$. *Then the following assertions are equivalent*:

(a) J *and* J' *are* 2-*equivalent*,

(b) $\mathcal{K}$ *and* $\mathcal{K}'$ *are isomorphic*,

(c) $f(\mathcal{K}) = f(\mathcal{K}')$.

PROOF. (a)$\to$(b). A 2-block of cardinality α in J is just the set of points of J which separate adjacent elements of a chain of length $\alpha + 1$ in $V \setminus J$. (See [1, Definition 4.2].) Thus saying that J and J' are 2-equivalent means that there is a length-preserving 1-1 correspondence between the chains of length > 1 of $V \setminus J$ and those of $V \setminus J'$. By [1, Theorem 5.7], the isomorphism type of $\mathcal{K}$ is determined by the lengths of the chains of length > 1 of $V \setminus J$. (See the remark following Theorem 5.7 in [1].) Therefore (a)$\to$(b).

(b)$\to$(a). Suppose $\mathcal{K} \approx \mathcal{B}(T^{\alpha_1} \oplus \cdots \oplus T^{\alpha_t})$, and $\mathcal{K}' \approx \mathcal{B}(T^{\beta_1} \oplus \cdots \oplus T^{\beta_u})$, where $1 \leqq \alpha_1 \leqq \cdots \leqq \alpha_t$ and $1 \leqq \beta_1 \leqq \cdots \leqq \beta_u$. By Theorem 5.9, $\mathcal{K}$ has t missing faces of cardinalities $1 + \alpha_1, \cdots, 1 + \alpha_t$, and $\mathcal{K}'$ has u missing faces, of cardinalities $1 + \beta_1, \cdots, 1 + \beta_u$, respectively. An isomorphism between $\mathcal{K}$ and $\mathcal{K}'$ maps mf $\mathcal{K}$ onto mf $\mathcal{K}'$. It follows that if $\mathcal{K}$ and $\mathcal{K}'$ are isomorphic, then $u = t$ and $\beta_i = \alpha_i$ for $1 \leqq i \leqq t$. This implies, by [1, Theorem 5.7], a length-preserving 1-1 correspondence between the chains of length > 1 of $V \setminus J$ and those of $V \setminus J'$, and therefore also a size-preserving 1-1 correspondence between the 2-blocks of J and those of J'. (See the first part of this proof.) Thus (b)$\to$(a).

(b)$\to$(c). Obvious.

(c)$\to$(b). $\mathcal{K}$ is an $(m-1)$-complex, and the missing faces of $\mathcal{K}$ are pairwise disjoint (see [1, Theorem 5.8]). The same holds for $\mathcal{K}'$. For $1 \leqq i \leqq m$, denote by r_i (r'_i) the number of missing i-faces, i.e., missing faces of cardinality $i + 1$, of $\mathcal{K}$ ($\mathcal{K}'$). Now assume $f(\mathcal{K}) = f(\mathcal{K}')$.

In order to prove that $\mathcal{K}$ and $\mathcal{K}'$ are isomorphic, it suffices to show that $r_i = r'_i$ for $1 \leqq i \leqq m$. This will be done by induction on i. Fix k, $1 \leqq k \leqq m$, and assume that $r_i = r'_i$ for $1 \leqq i < k$.

Then both $\mathcal{K}$ and $\mathcal{K}'$ have r_i missing i-faces for $1 \leqq i < k$. Since $\mathcal{K}$ and $\mathcal{K}'$ have the same number of vertices, and all missing faces of $\mathcal{K}$ (and of $\mathcal{K}'$) are pairwise disjoint, it follows that $\mathcal{K}$ and $\mathcal{K}'$ have isomorphic $(k-1)$-skeleta.

Define $\lambda_k(\mathcal{K})$ ($\lambda_k(\mathcal{K}')$) to be the number of $(k+1)$-subsets of vert $\mathcal{K}$ (vert $\mathcal{K}'$) which properly include a missing face of $\mathcal{K}$ ($\mathcal{K}'$). The isomorphism between $\mathrm{skel}_{k-1}(\mathcal{K})$ and $\mathrm{skel}_{k-1}(\mathcal{K}')$ implies $\lambda_k(\mathcal{K}) = \lambda_k(\mathcal{K}')$. Each $(k+1)$-subset of vert $\mathcal{K}$ is either a k-face of $\mathcal{K}$, or a missing k-face of $\mathcal{K}$, or properly includes a smaller missing face of $\mathscr{K}$, and these three possibilities are mutually exclusive. It follows that

$$\binom{f_0(\mathscr{K})}{k+1} = f_k(\mathscr{K}) + r_k + \lambda(\mathscr{K}),$$

and similarly

$$\binom{f_0(\mathscr{K}')}{k+1} = f_k(\mathscr{K}') + r'_k + \lambda_k(\mathscr{K}').$$

Since $f(\mathscr{K}) = f(\mathscr{K}')$ and $\lambda_k(\mathscr{K}) = \lambda_k(\mathscr{K}')$ it follows that $r_k = r'_k$.

This concludes the proof of the implication (c)→(b), and of Theorem 8.2.

9. Two lemmas

In this section we present two lemmas which will be used in the second part of the proof of Theorem 7.2 in Section 10. First we need some definitions.

All numbers appearing in this section are integers. A finite set I of integers is a *segment* if for all $a, c \in I$, $a < b < c$ implies $b \in I$. The segment $[a, b]$ is defined by $[a,b] = \{c : a \leqq c \leqq b\}$. Thus $[a,b] = \varnothing$ if $a \geqq b$.

Let S be a finite set of integers. A *block* of S is a segment in S which is maximal (with respect to inclusion).

A block B of S is even (odd) if it contains an even (odd) number of elements. Denote by $\nu(S)$ the number of odd blocks of S. Clearly $0 \leqq \nu(S) \leqq |S|$, and $\nu(S) \equiv |S| \pmod 2$. S is *separated* if $\nu(S) = |S|$. (Compare the definition of blocks in a cycle [1, Section 3].)

DEFINITION 9.1. For nonnegative integers $\alpha, \beta, \lambda, \mu$ let $G(\alpha, \beta, \lambda, \mu)$ be the set of all pairs of sets (S, T), such that $S \subset [1, \alpha]$, $T \subset [1, \beta]$, $|S| + |T| = \lambda$, and $\nu(S) + \nu(T) = \lambda - 2\mu$.

If one or more of the integers $\alpha, \beta, \lambda, \mu$ is negative, define $G(\alpha, \beta, \lambda, \mu) = \varnothing$. Define $g(\alpha, \beta, \lambda, \mu) = |G(\alpha, \beta, \lambda, \mu)|$.

Clearly $g(\alpha, \beta, \lambda, \mu) = 0$ if $\lambda < 2\mu$. If $S \subset [1, \alpha]$, then S has at least $\nu(S)$ blocks, and these blocks are separated by at least $\nu(S) - 1$ elements of $[1, \alpha] \setminus S$. Therefore $\alpha \geqq |S| + \nu(S) - 1$. Similarly, if $T \subset [1, \beta]$, then $\beta \geqq |T| + \nu(T) - 1$. Therefore, if $(S, T) \in G(\alpha, \beta, \lambda, \mu)$, then

$$\alpha + \beta \geqq |S| + |T| + \nu(S) + \nu(T) - 2 = 2\lambda - 2\mu - 2.$$

Thus

$$g(\alpha, \beta, \lambda, \mu) = 0 \qquad \text{if } \alpha + \beta < 2(\lambda - \mu - 1).$$

We shall show that the number $g(\alpha,\beta,\lambda,\mu)$ does not depend on the individual numbers α, β, but only on their sum, provided $\min(\alpha,\beta) \geqq \lambda-\mu$ (and even for $\min(\alpha,\beta)\geqq \lambda-1$, in case $\mu=0$ and $\lambda\geqq 1$). To show this, it clearly suffices to prove the following lemma.

LEMMA 9.2. *Suppose either* $\min(\alpha-1,\beta)\geqq\lambda-\mu$, *or* $\mu=0$, $\lambda\geqq 1$ *and* $\min(\alpha-1,\beta)\geqq\lambda-1$. *Then* $g(\alpha,\beta,\lambda,\mu)=g(\alpha-1,\beta+1,\lambda,\mu)$.

PROOF. First we shall establish an identity ((9.2.3) below) which will enable us to prove the lemma by induction on $\lambda-\mu$.

If $\alpha\geqq 2$, and $(S,T)\in G(\alpha,\beta,\lambda,\mu)$, then exactly one of the following three conditions holds:

(a) $\alpha\notin S$, in which case $(S,T)\in G(\alpha-1,\beta,\lambda,\mu)$,

(b) $\alpha\in S$ but $\alpha-1\notin S$, in which case $(S\setminus\{\alpha\},T)\in G(\alpha-2,\beta,\lambda-1,\mu)$,

(c) $\alpha\in S$ and $\alpha-1\in S$, in which case $(S\setminus\{\alpha-1,\alpha\},T)\in G(\alpha-2,\beta,\lambda-2,\mu-1)$.

Conversely, if $\alpha\geqq 2$, then $(S',T)\in G(\alpha-1,\beta,\lambda,\mu)$ implies $(S',T)\in G(\alpha,\beta,\lambda,\mu)$ and $\alpha\notin S'$; $(S',T)\in G(\alpha-2,\beta,\lambda-1,\mu)$ implies $(S'\cup\{\alpha\},T)\in G(\alpha,\beta,\lambda,\mu)$, $\alpha\in S'\cup\{\alpha\}$ and $\alpha-1\notin S'\cup\{\alpha\}$; $(S',T)\in G(\alpha-2,\beta,\lambda-2,\mu-1)$ implies $(S'\cup\{\alpha-1,\alpha\},T)\in G(\alpha,\beta,\lambda,\mu)$, and $\{\alpha-1,\alpha\}\in S'\cup\{\alpha-1,\alpha\}$.

It follows that for $\alpha\geqq 2$

$$g(\alpha,\beta,\lambda,\mu)=g(\alpha-1,\beta,\lambda,\mu)+g(\alpha-2,\beta,\lambda-1,\mu)+g(\alpha-2,\beta,\lambda-2,\mu-1). \tag{9.2.1}$$

Applying a similar argument to T we obtain, for $\alpha\geqq 1$, $\beta\geqq 1$, the identity

$$g(\alpha-1,\beta+1,\lambda,\mu)=g(\alpha-1,\beta,\lambda,\mu)+g(\alpha-1,\beta-1,\lambda-1,\mu)+g(\alpha-1,\beta-1,\lambda-2,\mu-1). \tag{9.2.2}$$

Subtraction of (9.1.2) from (9.1.1) yields, for $\alpha\geqq 2$, $\beta\geqq 1$:

$$\begin{aligned} &g(\alpha,\beta,\lambda,\mu)-g(\alpha-1,\beta+1,\lambda,\mu)\\ &\quad=-[g(\alpha-1,\beta-1,\lambda-1,\mu)-g(\alpha-2,\beta,\lambda-1,\mu)]\\ &\qquad-[g(\alpha-1,\beta-1,\lambda-2,\mu-1)-g(\alpha-2,\beta,\lambda-2,\mu-1)]. \end{aligned} \tag{9.2.3}$$

Now we proceed with the proof of our lemma.

Case I. $\mu=0$, $\lambda\geqq 1$. We assume $\min(\alpha-1,\beta)\geqq\lambda-1$ and show that $g(\alpha,\beta,\lambda,0)=g(\alpha-1,\beta+1,\lambda,0)$, by induction on λ.

Initial step: $\lambda = 1$. Then $\alpha \geqq 1$, and clearly $g(\alpha, \beta, 1, 0) = \alpha + \beta = g(\alpha - 1, \beta + 1, 1, 0)$.

Inductive step: Let $\lambda > 1$, and assume the statement holds for $\lambda - 1$. The assumption $\min(\alpha - 1, \beta) \geqq \lambda - 1$, implies $\min(\alpha - 2, \beta - 1) \geqq (\lambda - 1) - 1$, and therefore, by (9.2.3) and the induction hypothesis:

$$g(\alpha, \beta, \lambda, 0) - g(\alpha - 1, \beta + 1, \lambda, 0)$$
$$= -[g(\alpha - 1, \beta - 1, \lambda - 1, 0) - g(\alpha - 2, \beta, \lambda - 1, 0)] = 0.$$

((9.2.3) is applicable here, since $\alpha \geqq \lambda \geqq 2$, $\beta \geqq \lambda - 1 \geqq 1$.)

Case II. We assume $\min(\alpha - 1, \beta) \geqq \lambda - \mu$, and show that $g(\alpha, \beta, \lambda, \mu) = g(\alpha - 1, \beta + 1, \lambda, \mu)$, by induction on $\lambda - \mu$. If $\mu < 0$ or $\lambda < 2\mu$, then both sides of the above equality vanish. Assume therefore that $0 \leqq 2\mu \leqq \lambda$.

Initial step: If $\lambda - \mu = 0$, then $\lambda = \mu = 0$. But $g(\alpha, \beta, 0, 0) = g(\alpha - 1, \beta + 1, 0, 0) = 1$, since $\min(\alpha - 1, \beta) \geqq 0$.

Induction step: Let $\lambda - \mu \geqq 1$, and assume the lemma holds for all λ', μ' such that $\lambda' - \mu' < \lambda - \mu$. Since $\min(\alpha - 1, \beta) \geqq \lambda - \mu \geqq 1$, we have $\alpha \geqq 2$ and $\beta \geqq 1$ and may apply (9.2.3). Since $\min(\alpha - 2, \beta - 1) \geqq \lambda - \mu - 1 = (\lambda - 1) - \mu = (\lambda - 2) - (\mu - 1) < \lambda - \mu$, we conclude, by the induction hypothesis, that both brackets on the right-hand side of (9.2.3) vanish. □

For the second lemma we need another definition.

DEFINITION 9.3. For positive integers $j, l_1, \cdots, l_j$, let $\Gamma = \Gamma(l_1, \cdots, l_j)$ be a graph which is the disjoint union of j paths $L_1, \cdots, L_j$, where L_i is of length $l_i - 1$, i.e., L_i has l_i vertices, for $1 \leqq i \leqq j$.

For $\lambda \geqq 0$, let $G = G(l_1, \cdots, l_j, \lambda)$ be the collection of all independent sets of vertices of Γ of cardinality λ, which do not contain any end-vertex of Γ. Thus, an element of G is a set of λ mutually non-adjacent 2-valent vertices of Γ.

Finally, let $g(l_1, \cdots, l_j; \lambda)$ be the cardinality of $G(l_1, \cdots, l_j; \lambda)$.

LEMMA 9.4. *If* $j \geqq 2$ *and* $\min(l_1 - 1, l_2) \geqq \lambda + 1$, *then*

$$g(l_1, l_2, \cdots, l_j; \lambda) = g(l_1 - 1, l_2 + 1, l_3, \cdots, l_j; \lambda).$$

PROOF. If $\lambda = 0$, then both sides of the above equation equal 1. Therefore assume $\lambda \geqq 1$. Let W be the set of vertices of $L_3 \cup \cdots \cup L_j$. ($W = \varnothing$ if $j = 2$.) For $A \subset W$, let $G_A(l_1, \cdots, l_j; \lambda) = \{S \in G(l_1, \cdots, l_j; \lambda) : S \cap W = A\}$. Since $G(l_1, \cdots, l_j; \lambda)$ is the disjoint union of $G_A(l_1, \cdots, l_j; \lambda)$ over all subsets A of W, it suffices to show that the sets $G_A = G_A(l_1, l_2, \cdots, l_j; \lambda)$ and $G'_A = G_A(l_1 - 1, l_2 + 1, l_3, \cdots, l_j; \lambda)$ have the same cardinality for all $A \subset W$.

Now, if A is not independent, or if A contains an end-vertex of $L_3 \cup \cdots \cup L_j$, or if $|A| > \lambda$, then $G_A = G'_A = \emptyset$. If A is independent and contains no end-vertex of $L_3 \cup \cdots \cup L_j$ and $|A| = \lambda$, then $G_A = G'_A = \{A\}$. In the remaining case we have $|A| < \lambda$, and therefore $\lambda > 0$. Using the notation of Definition 9.1 we obtain in this case:

$$|G_A| = g(l_1 - 2, l_2 - 2, \lambda - |A|, 0), \qquad |G'_A| = g(l_1 - 3, l_2 - 1, \lambda - |A|, 0).$$

These numbers are well defined, since $\min(l_1 - 1, l_2) \geqq \lambda + 1 \geqq 2$ implies $\min(l_1 - 3, l_2 - 2) \geqq \lambda - 1 \geqq 0$. Since $0 < \lambda - |A| \leqq \lambda$, we can apply Lemma 9.2 with $\mu = 0$ and obtain $|G_A| = |G'_A|$. □

10. Proof of Theorem 7.2, part II: shifting blocks

Let J, J' be separated j-subsets of V, where $V = [1, v]$, $0 \leqq j < m$, $v \geqq 2m + 3 \geqq 5$, and let $\alpha = m - j + 2$. We have to show that if J and J' are α-equivalent, then $f(\mathscr{C}(V, 2m)/J) = f(\mathscr{C}(V, 2m)/J')$. If $j \leqq 1$, or if J and J' consist of a single α-block, then J and J' are similar, in the sense of Definition 7.1(a), and the corresponding quotients are isomorphic.

Assume therefore that $j \geqq 2$, and that J consists of at least two α-blocks. Let $B_1, \cdots, B_q$ be the α-blocks of J, and suppose they appear in this order on $C(V)$. For $1 \leqq i \leqq q$, denote by D_i the segment of $C(V)$ that separates B_i from B_{i+1} (where $B_{q+1} = B_1$). Clearly $|D_i| \geqq \alpha$. Then J' can be obtained from J by a finite sequence of steps of four types:

I. Shift B_i forward by one unit, provided $|D_i| > \alpha$. This will increase $|D_{i-1}|$ by 1, and decrease $|D_i|$ by 1.

II. Reflect B_i about its midpoint. This leaves D_{i-1} and D_i unchanged.

III. Exchange the position of B_i and B_{i+1}. This may cause a shift in D_i, but will leave $|D_i|$ unchanged.

IV. Apply to J an automorphism (rotation or reflection) of $C(V)$.

A step of type IV clearly preserves the isomorphism type of $\mathscr{C}(V, 2m)/J$, and therefore its f-vector.

In this section we shall show that a step of type I does not affect the f-vector. The corresponding result for steps of type II and III will be proved in the next section.

Since the isomorphism type of $\mathscr{C}(V, 2m)/J$ is preserved under rotations of J in $C(V)$, we may assume that $v \in J$. This will somewhat simplify the notation. For the formal statement of our result we shall need the following definition. (See Fig. 3.)

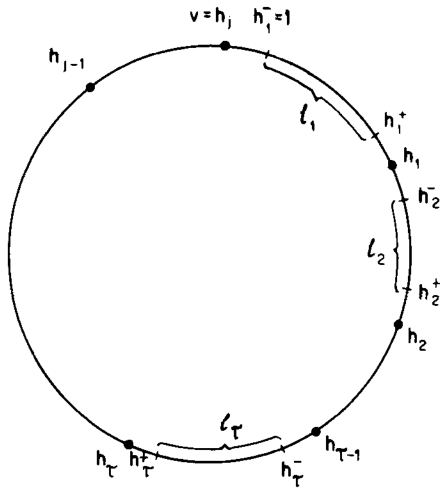


Fig. 3.

Definition 10.1. Given positive integers v, m, j, $l_1, \cdots, l_j$, such that $j \leqq m$ and $v = j + \Sigma_{i=1}^{j} l_i = \Sigma_{i=1}^{j}(1 + l_i) \geqq 2m + 3$, define, for $0 \leqq k \leqq j : h_k = \Sigma_{i=1}^{k}(1 + l_i)$. Then $0 = h_0 < h_1 < \cdots < h_j = v$, and $h_k - h_{k-1} = 1 + l_k \geqq 2$ for $1 \leqq k \leqq j$. Define also: $V = [1, v]$, $J = \{h_1, \cdots, h_j\}$, $\mathcal{K} = \mathcal{K}(m; l_1, \cdots, l_j) = \mathcal{C}(V, 2m)/J$, $f_k(m; l_1, \cdots, l_j) = f_k(\mathcal{K})$ $(0 \leqq k < 2m - j)$ and $f(m; l_1, \cdots, l_j) = f(\mathcal{K}) = (f_0(\mathcal{K}), f_1(\mathcal{K}), \cdots, f_{2m-j-1}(\mathcal{K}))$.

Note that, for $1 \leqq i \leqq j$, h_{i-1} and h_i belong to the same α-block of J iff either $l_i < \alpha$, or all the numbers $l_k, k \neq i$, are $< \alpha$.

The vector $f(m; l_1, \cdots, l_j)$ is clearly invariant under cyclic permutations and reflections of the sequence $(l_1, \cdots, l_j)$.

Definition 10.2. Two sequences $(l_1, \cdots, l_j)$ and $(l'_1, \cdots, l'_j)$ of positive integers are *α-equivalent* if $\Sigma_{i=1}^{j} l_i = \Sigma_{i=1}^{j} l'$, and $\min(l_i, \alpha) = \min(l'_i, \alpha)$ (i.e., $l_i = l'_i$ or $\min(l_i, l'_i) \geqq \alpha$) for $1 \leqq i \leqq j$.

Note that the sequences $(l_1, \cdots, l_j)$ and $(l'_1, \cdots, l'_j)$ are α-equivalent iff the corresponding sets J, J' (see Definition 10.1) are obtained from each other by a finite sequence of shifts of α-blocks, which leave the point $v \in V$ fixed. These shifts, or their inverses, are steps of type I, as described above.

Theorem 10.3. *Suppose v, m, j, $l_1, \cdots, l_j$ are as in Definition* 10.1, *and* $2 \leqq j < m$. *If $(l_1, \cdots, l_j)$ and $(l'_1, \cdots, l'_j)$ are $(m - j + 2)$-equivalent, then* $f(m; l_1, \cdots, l_j) = f(m; l'_1, \cdots, l'_j)$.

Theorem 10.3 is an immediate consequence of the following theorem:

THEOREM 10.4. *Suppose* m, j, $l_1, \cdots, l_j$ *are positive integers satisfying* $j + \sum_{i=1}^{j} l_i \geqq 2m + 3 \geqq 2j + 3 \geqq 7$. *If* $\sigma, \tau \in [1, j]$, $\sigma \neq \tau$, *and* $l'_i = l_i - \delta_{i\sigma} + \delta_{i\tau}$ *for* $1 \leqq i \leqq j$ (*i.e.*, $l'_\sigma = l_\sigma - 1$, $l'_\tau = l_\tau + 1$, $l'_i = l_i$ *for* $i \in [1, j] \setminus \{\sigma, \tau\}$), *and if* $\min(l'_\sigma, l_\tau) \geqq m - j + 2$, *then* $f(m; l_1, \cdots, l_j) = f(m; l'_1, \cdots, l'_j)$.

(Theorem 10.3 clearly follows from Theorem 10.4, by induction on $\sum_{i=1}^{j} | l'_i - l_i |$.)

PROOF. Our theorem holds for $j = m$, by Theorem 8.2. We shall therefore restrict our attention to the cases where $2 \leqq j < m$.

We shall use the notation introduced in Definition 10.1, and denote by h'_k, J', $\mathcal{K}'$ the entities which correspond to h_k, J and $\mathcal{K}$ relative to the sequence $(l'_1, \cdots, l'_j)$.

Since $\mathcal{C}(V, 2m)$ is m-neighborly (i.e., every m vertices determine a face), with v vertices, $\mathcal{K} = \mathcal{C}(V, 2m)/J$ is $(m - j)$-neighborly, with $v - j$ vertices, and therefore $f_k(\mathcal{K}) = \binom{v-j}{k+1}$ for $0 \leqq k < m - j$. The same holds for $\mathcal{K}'$ (see also [1, Theorem 4.5]). Moreover, $f(\mathcal{K})$, $f(\mathcal{K}')$ are f-vectors of simplicial $(2m - j)$-polytopes, and therefore satisfy the Dehn–Sommerville equations (E_k^{2m-j}) $-1 \leqq k \leqq 2m - j - 2$ (see [2, p. 146]). These equations determine the upper half of $f(\mathcal{K})$ (and of $f(\mathcal{K}')$) in terms of its lower half. More precisely, for $t = [\frac{1}{2}(2m - j)]$, the numbers $f_t(\mathcal{K}), \cdots, f_{2m-j-1}(\mathcal{K})$ are linear functions of the numbers $f_{-1}(\mathcal{K})$ $(= 1)$, $f_0(\mathcal{K}), \cdots, f_{t-1}(\mathcal{K})$.

In order to establish the equality of $f(\mathcal{K})$ and $f(\mathcal{K}')$, it therefore suffices to show that $f_k(\mathcal{K}) = f_k(\mathcal{K}')$ for

$$m - j \leqq k \leqq [\tfrac{1}{2}(2m - j)] - 1 = m - \left[\frac{j+3}{2}\right]$$

Thus, for $j = 2, 3$ the only interesting value of k is $k = m - j$.

In the proof of $f_k(\mathcal{K}) = f_k(\mathcal{K}')$ for general k, which will be rather complicated, we shall find it convenient to assume that $4 \leqq j$. Therefore we shall first give a separate (and simpler) proof of $f_{m-j}(\mathcal{K}) = f_{m-j}(\mathcal{K}')$.

Since $\mathcal{K}$ is $(m - j)$-neighborly, every $m - j + 1$ vertices of $\mathcal{K}$ determine either an $(m - j)$-face or a missing face (see [1, Definition 2.1]); the same for $\mathcal{K}'$. Thus $f_{m-j}(\mathcal{K}) = f_{m-j}(\mathcal{K}')$ iff $\mathcal{K}$ and $\mathcal{K}'$ have the same number of missing faces with $m - j + 1$ vertices. By [1, Theorem 4.5] (see also [1, Definition 4.2]), an $(m - j + 1)$-subset S of $V \setminus J$ is a missing face of $\mathcal{K}$ iff no element of S is adjacent (in $C(V)$) to an element of J, and no two elements of S are adjacent to each other. Since $C(V) \setminus J$ is the disjoint union of j paths of lengths $l_1 - 1, \cdots, l_j - 1$, it is a graph of type $\Gamma(l_1, \cdots, l_j)$, as defined in Definition 9.3, and the set of missing faces of $\mathcal{K}$ with $m-j+1$ vertices is precisely the collection $G(l_1,\cdots,l_j\,;m-j+1)$ defined there.

Applying Lemma 9.4 with $\lambda = m-j+1$, we find that

$$|G(l_1,\cdots,l_j\,;m-j+1)| = |G(l'_1,\cdots,l'_j;m-j+1)|,$$

and therefore $f_{m-j}(\mathcal{K}) = f_{m-j}(\mathcal{K}')$.

From now on we assume that $4 \leqq j < m$.

Since the number $f_k(\mathcal{K}) = f_k(m\,;l_1,\cdots,l_j)$ is invariant under cyclic permutations (and reflections) of the sequence $(l_1,\cdots,l_j)$, we shall also assume, without loss of generality, that $\sigma = 1$.

At this point we introduce some further notations, as follows: For $1 \leqq i \leqq j$, let $h_i^- = h_{i-1}+1$, $h_i^+ = h_i - 1$, $I_i = [h_i^-, h_i^+]$, $I_i^0 = [h_i^- + 1, h_i^+ - 1]$. (See Fig. 3. Recall that by Definition 10.1, $h_0 = 0$, $h_j = v$, $J = \{h_1,\cdots,h_j\}$, and note that $|I_i| = l_i$, and $I_i^0 = \varnothing$ if $l_i \leqq 2$.)

Denote by $\mathcal{F}_k$ the set of k-faces of $\mathcal{K}$, and define subsets of $\mathcal{F}_k$ as follows:

$$\mathcal{F}_k(1) = \{\Phi \in \mathcal{F}_k : h_1^- \in \Phi\}, \qquad \mathcal{F}_k(2) = \{\Phi \in \mathcal{F}_k : h_1^+ \in \Phi\},$$

$$\mathcal{F}_k(3) = \{\Phi \in \mathcal{F}_k : h_\tau^- \in \Phi\}, \qquad \mathcal{F}_k(4) = \{\Phi \in \mathcal{F}_k : h_\tau^+ \in \Phi\},$$

$$\mathcal{F}_k(0) = \mathcal{F} \setminus \bigcup_{i=1}^{4} \mathcal{F}_k(i);$$

and for every non-empty subset S of $\{1,2,3,4\}$:

$$\mathcal{F}_k(S) = \bigcap\{\mathcal{F}_k(i) : i \in S\}.$$

Then clearly

$$\begin{aligned} f_k(\mathcal{K}) &= |\mathcal{F}_k| = |\mathcal{F}_k(0)| + \left|\bigcup_{i=1}^{4} \mathcal{F}_k(i)\right| \\ &= |\mathcal{F}_k(0)| + \sum \{(-1)^{|S|-1}|\mathcal{F}_k(S)| : \varnothing \neq S \subset \{1,2,3,4\}\}. \end{aligned} \tag{10.4.1}$$

The entities corresponding to $\mathcal{F}_k$, $\mathcal{F}_k(i)$, $\mathcal{F}_k(S)$, h_i^-, h_i^+, I_i, I_i^0 for the sequence $(l'_1,\cdots,l'_j)$ will be denoted, quite naturally, by $\mathcal{F}'_k$, $\mathcal{F}'_k(i)$, $\mathcal{F}'_k(S)$, etc. From equation (10.4.1), and its dashed counterpart, it follows that in order to prove that $f_k(\mathcal{K}) = f_k(\mathcal{K}')$ it suffices to show:

(a) $|\mathcal{F}_k(0)| = |\mathcal{F}'_k(0)|$, and

(b) $|\mathcal{F}_k(S)| = |\mathcal{F}'_k(S)|$ for all $\varnothing \neq S \subset \{1,2,3,4\}$.

We shall establish (a) directly; (b) will be proved by an inductive argument, using the assumption that Theorem 10.4 holds for smaller values of the parameter m.

PROOF OF (a). If A is a set of integers and b is an integer, define $A+b=\{a+b : a \in A\}$. For $1<i\leqq j$, $i\neq\tau$, and $A\subset I_i=[h_i^-,h_i^+]$, define $A'=A-h_i+h_i'$ (i.e., $A'=A-1$ if $1<i<\tau$, $A'=A$ if $\tau<i\leqq j$). Then $A\rightarrow A'$ is a 1-1 correspondence between the subsets of I_i and those of I_i'. If $A\subset V\backslash(J\cup I_1\cup I_\tau)$, define $A'=\bigcup\{(A\cap I_i)' : i\in[2,j]\backslash\{\tau\}\}$, i.e., $A'=((A\cap[h_1,h_{\tau-1}])-1)\cup(A\cap[h_\tau,v])$. Then clearly $|A'|=|A|$, and the mapping $A\rightarrow A'$ is a 1-1 correspondence between the subsets of $V\backslash(J\cup I_1\cup I_\tau)$ and those of $V\backslash(J'\cup I_1'\cup I_\tau')$. Moreover,

$$A'\cup J'=(((A\cup J)\cap[h_1,h_{\tau-1}])-1)\cup((A\cup J)\cap[h_\tau,v]),$$

and $\nu(A'\cup J')=\nu(A\cup J)$. (Here $\nu(S)$ is the number of odd blocks of S, as defined at the beginning of Section 9.)

For $A\subset V\backslash(J\cup I_1\cup I_\tau)$ define $\mathcal{F}_k(0,A)=\{\Phi\in\mathcal{F}_k(0):\Phi\backslash(I_1\cup I_\tau)=A\}$, i.e., $\mathcal{F}_k(0,A)$ is the set of all k-faces Φ of $\mathcal{K}$ such that $\Phi\cap\{h_1^-,h_1^+,h_\tau^-,h_\tau^+\}=\varnothing$ and $\Phi\backslash(I_1\cup I_\tau)=A$. Similarly we define $\mathcal{F}_k'(0,B)$ for $B\subset V\backslash(J\cup I_1'\cup I_\tau')$.

Now $\mathcal{F}_k(0)$ is the disjoint union of the sets $\mathcal{F}_k(0,A)$ where A ranges over all subsets of $V\backslash(J\cup I_1\cup I_2)$, and $\mathcal{F}_k'(0)$ is the disjoint union of the corresponding sets $\mathcal{F}_k'(0,A')$. Therefore, in order to prove (a) it suffices to show that $|\mathcal{F}_k(0,A)|=|\mathcal{F}_k'(0,A')|$ for all $A\subset V\backslash(J\cup I_1\cup I_\tau)$. We shall show this now, using Lemma 9.2.

Every member of $\mathcal{F}_k(0,A)$ can be represented uniquely as a union $A\cup S\cup T$, where $S\subset I_1^0$, $T\subset I_\tau^0$. Such a union $A\cup S\cup T$ is a k-face of $\mathcal{K}$ iff $|A\cup S\cup T|=k+1$, and $J\cup A\cup S\cup T$ is a face of $\mathcal{C}(V,2m)$, i.e., iff (see [1, Theorem 3.3])

$$\begin{aligned}&|A\cup S\cup T|=k+1 \quad\text{and}\\ &|J\cup A\cup S\cup T|+\nu(J\cup A\cup S\cup T)\leqq 2m.\end{aligned}\tag{10.4.2}$$

The three sets $A\cup J$, S and T are pairwise disjoint. Moreover, no element of any one of them is adjacent to elements of the other two sets. Therefore

$$\begin{aligned}&|A\cup S\cup T|=|A|+|S|+|T|, \quad\text{and}\\ &\nu(A\cup J\cup S\cup T)=\nu(A\cup J)+\nu(S)+\nu(T).\end{aligned}\tag{10.4.3}$$

We may assume that $|A|\leqq k+1$, since otherwise $\mathcal{F}_k(0,A)=\mathcal{F}_k'(0,A')=\varnothing$.

Define: $\lambda=k+1-|A|$. Then $A\cup S\cup T\in\mathcal{F}_k(0,A)$ implies $|S|+|T|=\lambda$. Choose sets $S\subset I_1^0$, $T\subset I_\tau^0$, such that $|S|+|T|=\lambda$ (if such a choice is at all possible), and define $\mu=\frac{1}{2}(\lambda-\nu(S)-\nu(T))$. Then μ is an integer, and $\nu(S)+\nu(T)=\lambda-2\mu$. From (10.4.2) and (10.4.3) it follows that $A\cup S\cup T\in\mathcal{F}_k(0,A)$ iff

$$2m \geqq j + k + 1 + \nu(J \cup A) + \nu(S) + \nu(T) = j + k + 1 + \nu(J \cup A) + \lambda - 2\mu,$$

i.e., iff $\mu \geqq \mu^*$, where $\mu^* = \frac{1}{2}(j + k + 1 + \lambda + \nu(J \cup A)) - m$. Note that μ^* is an integer, since $\nu(J \cup A) \equiv |J \cup A| \pmod 2$, and $|J \cup A| = |J| + |A| = j + k + 1 - \lambda$. Note also that $\nu(J) = |J|$, since J is separated. Adding the points of A to J one by one will change the number of odd blocks by exactly one at each step. Therefore $\nu(J \cup A) \geqq \nu(J) - |A| = j - |A|$. It follows that

$$\begin{aligned}\mu^* &\geqq \tfrac{1}{2}(j + k + 1 + \lambda + j - |A|) - m \\ &= \tfrac{1}{2}(j + k + 1 + \lambda + j - (k + 1 - \lambda)) - m = j + \lambda - m.\end{aligned}$$

The number of pairs (S, T), $S \subset I_1^0$, $T \subset I_\tau^0$, such that $|S| + |T| = \lambda$ and $\nu(S) + \nu(T) = \lambda - 2\mu$ is exactly $g(\alpha, \beta, \lambda, \mu)$ (see Definition 9.1), where $\alpha = |I_1^0| = l_1 - 2$, $\beta = |I_\tau^0| = l_\tau - 2$. Thus

$$(10.4.4) \qquad |\mathcal{F}_k(0, A)| = \sum \left\{ g(l_1 - 2, l_\tau - 2, \lambda, \mu) : \max(0, \mu^*) \leqq \mu \leqq \left[\frac{\lambda}{2}\right] \right\},$$

where $\lambda = k + 1 - |A|$, and $\mu^* = \frac{1}{2}(j + k + 1 + \lambda - \nu(J \cup A)) - m \geqq j + \lambda - m$.

For $|\mathcal{F}'_k(0, A')|$ we obtain a similar expression, with the same values of λ and μ^*, since $|A'| = |A|$, and $\nu(J' \cup A') = \nu(J \cup A)$, i.e.,

$$(10.4.5) \qquad |\mathcal{F}'_k(0, A')| = \sum \left\{ g(l'_1 - 2, l'_\tau - 2, \lambda, \mu) : \max(0, \mu^*) \leqq \mu \leqq \left[\frac{\lambda}{2}\right] \right\}.$$

The above sum may be empty, but then the corresponding sum in (10.4.4) is also empty, and both vanish.

Writing $\alpha = l_1 - 2$, $\beta = l_\tau - 2$, we obtain $g(l_1 - 2, l_\tau - 2, \lambda, \mu) = g(\alpha, \beta, \lambda, \mu)$ and $g(l'_1 - 2, l'_\tau - 2, \lambda, \mu) = g(\alpha - 1, \beta + 1, \lambda, \mu)$. These two numbers are equal, by Lemma 9.2, provided $\min(\alpha - 1, \beta) \geqq \lambda - \mu$. By the assumptions of Theorem 10.4, $\min(\alpha - 1, \beta) = \min(l'_1, l_\tau) - 2 \geqq m - j$. On the other hand, since $\mu \geqq \mu^*$, we have $\lambda - \mu \leqq \lambda - \mu^* \leqq \lambda - (j + \lambda - m) = m - j$. It follows that $g(l_1 - 2, l_\tau - 2, \lambda, \mu) = g(l'_1 - 2, l'_\tau - 2, \lambda, \mu)$ for $\mu \geqq \mu^*$, and therefore $|\mathcal{F}_k(0, A)| = |\mathcal{F}'_k(0, A')|$. This proves (a).

Proof of (b). We have to show that $|\mathcal{F}_k(S)| = |\mathcal{F}'_k(S)|$ for $\emptyset \neq S \subset \{1, 2, 3, 4\}$. We asume that $4 \leqq j < m$. (The cases $j \leqq 3$ and $j = m$ have been settled already.)

Case I: $S = \{1\}$. $\Phi \in \mathcal{F}_k(S)$ iff Φ is a k-face of $\mathcal{K}$ $(= \mathcal{C}(V, 2m)/J)$ and $h_1^- = 1 \in \Phi$, i.e., iff Φ is a $(k+1)$-subset of $V \setminus J$, $1 \in \Phi$, and $\Phi \cup J$ is a face of $\mathcal{C}(V, 2m)$. Therefore $|\mathcal{F}_k(\{1\})|$ is equal to the number of k-subsets Ψ of $V \setminus (J \cup \{1\})$, such that $\Psi \cup (J \cup \{1\})$ is a face of $\mathcal{C}(V, 2m)$, i.e., $|\mathcal{F}_k(S)| = f_{k-1}(\mathcal{C}(V, 2m)/(J \cup \{1\}))$.

Since $v \in J$, we obtain, by [1, Theorem 3.5],

$$\begin{aligned}\mathscr{C}(V,2m)/(J\cup\{1\}) &= \mathscr{C}(V\setminus\{v,1\},2m-2)/(J\setminus\{v\})\\ &\approx \mathscr{K}(m-1;l_j+l_1-1,l_2,\cdots,l_{j-1})\end{aligned}$$

(see Definition 10.1; " $\approx$ " means "is isomorphic to"). Thus

$$|\mathscr{F}_k(S)| = f_{k-1}(m-1;l_j+l_1-1,l_2,\cdots,l_j-1). \tag{10.4.6}$$

In the same manner we obtain

$$|\mathscr{F}'_k(S)| = f_{k-1}(m-1;l'_j+l'_1-1,l'_2,\cdots,l'_{j-1}). \tag{10.4.7}$$

If $\tau = j$, then the right-hand sides of (10.4.6) and (10.4.7) are identical.

If $\tau < j$, then we use the inductive assumption that Theorem 10.4 holds when m is replaced by $m-1$. Applying Theorem 10.4 with m, j, k, σ, l_σ replaced by $m-1$, $j-1$, $k-1$, 1, l_j+l_1-1, respectively, we conclude that the right-hand sides of (10.4.6) and (10.4.7) are equal, since $\min(l'_j+l'_1-1,l_\tau) \geqq \min(l'_1,l_\tau) \geqq m-j+2=(m-1)-(j-1)+2$. Therefore $|\mathscr{F}_k(S)| = |\mathscr{F}'_k(S)|$.

The same type of argument will also settle the cases $S=\{2\}$, $S=\{3\}$ and $S=\{4\}$. Besides, these three cases can also be transformed into Case I by suitable rotations or reflections of $C(V)$.

Case II: $S=\{1,2\}$. Since $|\mathscr{F}_k(S)| = |\mathscr{F}'_k(S)| = 0$ for $k<1$, and $=1$ for $k=1$, we shall assume that $k \geqq 2$.

Using the same argument as in the beginning of Case I, we find that $|\mathscr{F}_k(S)|$ is equal to the number of $(k-1)$-subsets Ψ of $V\setminus(J\cup\{1,h_1^+\})$, such that $\Psi\cup(J\cup\{1,h_1^+\})$ is a face of $\mathscr{C}(V,2m)$, i.e., $|\mathscr{F}_k(S)| = f_{k-2}(\mathscr{C}(V,2m)/(J\cup\{1,h_1^+\}))$. Since $v\in J$ and $h_1=h_1^++1\in J$, $\mathscr{C}(V,2m)/(J\cup\{1,h_1^+\})$ is equal to

$$\mathscr{C}(V\setminus\{v,1,h_1^+,h_1\},2(m-2))/(J\setminus\{v,h_1\}) \approx \mathscr{K}(m-2,l_j+l_1+l_2-2,l_3,\cdots,l_{j-1}).$$

Therefore $|\mathscr{F}_k(S)| = f_{k-2}(m-2;l_j+l_1+l_2-2,l_3,\cdots,l_{j-1})$. Similarly

$$|\mathscr{F}'_k(S)| = f_{k-2}(m-2;l'_j+l'_1+l'_2-2,l'_3,\cdots,l'_{j-1}).$$

If $\tau=2$ or $\tau=j$, then the expressions on the right-hand side of the last two equations are identical. If $2<\tau<j$, then the equality of these two expressions follows from the inductive hypothesis that Theorem 10.4 holds with m, j, k, σ, l_σ, τ replaced by $m-2$, $j-2$, $k-2$, 1, $l_j+l_1+l_2-2$, $\tau-1$, respectively. Note that $\min(l_j+l'_1+l_2-2,l_\tau) \geqq \min(l'_1,l_\tau) \geqq m-j+2=(m-2)-(j-2)+2$. Therefore $\mathscr{F}_k(S)=\mathscr{F}'_k(S)$.

The case $S=\{3,4\}$ can be reduced to the case $S=\{1,2\}$ by a suitable rotation of $C(V)$.

Case III: $S = \{1,3\}$ $(k \geqq 2)$. The same arguments as before show that

$$|\mathcal{F}_k(S)| = f_{k-2}(\mathscr{C}(V\setminus\{v,1,h_{\tau-1},h_\tau^-\},2(m-2))/(J\setminus\{v,h_{\tau-1}\}))$$

$$= \begin{cases} f_{k-2}(m-2; l_j + l_1 + l_2 - 2, l_3, \cdots, l_{j-1}) & \text{if } \tau = 2 \\ f_{k-2}(m-2; l_{j-1} + l_j + l_1 - 2, l_2, \cdots, l_{j-2}) & \text{if } \tau = j \\ f_{k-2}(m-2; l_j + l_1 - 1, l_2, \cdots, l_{\tau-2}, l_{\tau-1} + l_\tau - 1, l_{\tau+1}, \cdots, l_{j-1}) & \\ & \text{if } 2 < \tau < j. \end{cases}$$

For $|\mathcal{F}'_k(S)|$ we obtain the same expressions with $l_1, \cdots, l_j$ replaced by $l'_1, \cdots, l'_j$.

If $\tau = 2$ or $\tau = j$, then the expressions for $|\mathcal{F}_k(S)|$ and $|\mathcal{F}'_k(S)|$ are identical. If $2 < \tau < j$, use Theorem 10.4 (the induction hypothesis) with m, j, k, σ, l_σ, τ, l_τ replaced by $m-2, j-2, k-2, 1, l_j + l_1 - 1, \tau - 1, l_{\tau-1} + l_\tau - 1$, respectively.

The case $S = \{2,4\}$ is treated in the same way.

Case IV: $S = \{2,3\}$ $(k \geqq 2)$. For $\tau > 3$, this case is treated in the same manner as Case III.

For $\tau = 3$, we obtain $|\mathcal{F}_k(S)| = |\mathcal{F}'_k(S)| = f_{k-2}(m-2; l_1 + l_2 + l_3 - 2, l_4, \cdots, l_j)$.

If $\tau = 2$, then a k-face Φ of $\mathcal{H}$ belongs to $\mathcal{F}_k(S)$ iff $h_1^+(= h_1 - 1)$ and $h_2^-(= h_1 + 1)$ belong to Φ. Therefore $|\mathcal{F}_k(S)| = f_{k-2}(\mathscr{C}(V,2m)/(J \cup \{h_1^+, h_2^-\}))$. (For details, see Case I.)

By [1, Theorem 3.5],

$$\mathscr{C}(V,2m)/(J \cup \{h_1^+,h_2^-\}) = \mathscr{C}(V\setminus\{h_1^+,h_1\},2(m-1))/((J\setminus\{h_1\}) \cup \{h_2^-\})$$

$$\approx \mathcal{H}(m-1; l_1 - 1, l_2 - 1, l_3, \cdots, l_j).$$

Therefore $|\mathcal{F}_k(S)| = f_{k-2}(m-1; l_1 - 1, l_2 - 1, l_3, \cdots, l_j)$, and similarly $|\mathcal{F}'_k(S)| = f_{k-2}(m-1; l'_1 - 1, l'_2 - 1, l_3, \cdots, l_j)$. Since we are assuming that $j < m$ (see the opening sentence of the proof of Theorem 10.4), we find that

$$f_{k-2}(m-1; l_1 - 1, l_2 - 1, l_3, \cdots, l_j) = f_{k-2}(m-1; l'_1 - 1, l'_2 - 1, l_3, \cdots, l_j)$$

by Theorem 10.4 (the induction hypothesis), with m, k, σ, l_σ, τ, l_τ replaced by $m-1$, $k-2$, 1, $l_1 - 1$, 2, $l_2 - 1$, respectively (j remains unchanged). Note that $\min(l'_1 - 1, l_2 - 1) = \min(l'_1, l_2) - 1 \geqq (m - j + 2) - 1 = (m-1) - j + 2$. Therefore $|\mathcal{F}_k(S)| = |\mathcal{F}'_k(S)|$.

The case $S = \{1,4\}$ is treated in the same manner.

Case V: $S = \{1,2,3\}$ $(k \geqq 2)$. Here we distinguish four subcases.

V_1. $3 < \tau < j$. As before, we obtain:

$$|\mathcal{F}_k(S)| = f_{k-3}(m-3; l_j + l_1 + l_2 - 2, l_3, \cdots, l_{\tau-2}, l_{\tau-1} + l_\tau - 1, l_{\tau+1}, \cdots, l_{j-1}),$$

and a similar expression for $|\mathcal{F}'_k(S)|$. The desired equality follows from Theorem 10.4 (the induction hypothesis) with m, k, j, σ, l_σ, τ, l_τ replaced by $m-3$, $k-3$, $j-3$, 1, $l_j+l_1+l_2-2$, $\tau-1$, $l_{\tau-1}+l_\tau-1$, respectively.

V_2. $\tau=3$. In this subcase we obtain for $|\mathcal{F}_k(S)|$ and $|\mathcal{F}'_k(S)|$ the same expression, namely $f_{k-3}\ (m-3; l_j+l_1+l_2+l_3-3, l_4, \cdots, l_{j-1})$.

V_3. $\tau=2$. Here we obtain, as in Case IV (with $\tau=2$):

$$|\mathcal{F}_k(S)| = f_{k-3}(m-2; l_j+l_1-2, l_2-1, l_3, \cdots, l_{j-1}).$$

For $|\mathcal{F}'_k(S)|$ we obtain a similar expression, with l_1, l_2 replaced by l'_1, l'_2. These two expressions are equal by Theorem 10.4 (the induction hypothesis), with m, k, j, σ, l_σ, τ, l_τ replaced by $m-2$, $k-3$, $j-1$, 1, l_j+l_1-2, 2, l_2-1, respectively. Notice that the reduced values of the parameters remain within the scope of Theorem 10.4, since we assume that $4 \leqq j < m$.

V_4. $\tau=j$. As in subcase V_2, we obtain for $|\mathcal{F}_k(S)|$ and $|\mathcal{F}'_k(S)|$ the same expression, namely $f_{k-3}(m-3; l_{j-1}+l_j+l_1+l_2-3, l_3, \cdots, l_{j-2})$.

This concludes Case V. The cases $S=\{1,2,4\}$, $S=\{1,3,4\}$ and $S=\{2,3,4\}$ can be treated in the same manner. They can also be reduced to Case V by suitable rotations or reflections of $C(V)$.

Case VI: $S=\{1,2,3,4\}$ $(k \geqq 3)$. Here we distinguish six subcases.

VI_1. $3<\tau<j-1$. Proceed as in subcase V_1 above.

VI_2. $3=\tau<j-1$. Proceed as in subcase V_2 above.

VI_3. $3<\tau=j-1$. Same as VI_2.

VI_4. $3=\tau=j-1$. Here $j=4$, and we obtain: $|\mathcal{F}_k(S)|=|\mathcal{F}'_k(S)|= f_{k-4}(C(v-8, 2(m-4)))$.

VI_5. $\tau=2$. Proceed as in subcase V_3 above. The expression obtained for $|\mathcal{F}_k(S)|$ is $f_{k-4}(m-3; l_j+l_1-2, l_2+l_3-2, l_3, \cdots, l_{j-1})$, and the induction hypothesis is used with m, k, j, σ, l_σ, τ, l_τ replaced by $m-3$, $k-4$, $j-2$, 1, l_j+l_1-2, 2, l_2+l_3-2, respectively.

VI_6. $\tau=j$. Same as VI_5.

This concludes Case VI, and with it the proof of (b), and with it the proof of Theorem 10.4.

REMARK. We could have proved assertion (b) ($|\mathcal{F}_k(S)|=|\mathcal{F}'_k(S)|$ for all $\emptyset \neq S \subset \{1,2,3,4\}$) without the restriction $j \geqq 4$. This would save us the separate proof of Theorem 10.4 for $j=2,3$, would render Lemma 9.4 superfluous, and would make the whole proof of Theorem 10.4 independent of the Dehn–Sommerville equations: all this at the cost of a few extra cases in the inductive proof of assertion (b).

11. Proof of Theorem 7.2, part III: reflections

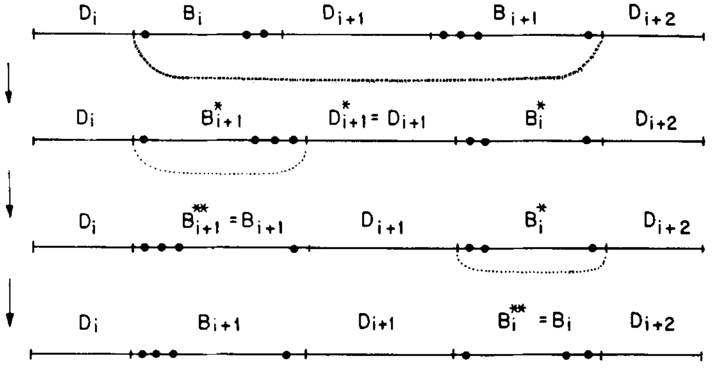
As in the opening paragraph of Section 10, let J, J' be separated j-subsets of V ($V = [1, v]$, $2 \leqq j < m$, $v \geqq 2m + 3$) and let $\alpha = m - j + 2$. We observed that if J and J' are α-equivalent, then J can be transformed into J' by a finite sequence of steps of four types, as described in the beginning of Section 10. From Theorem 10.3 it follows that the f-vector of $\mathscr{C}(V, 2m)/J$ remains unchanged when J undergoes a transformation of type I. The same holds trivially for transformations of type IV. In this section we shall prove that transformations of types II and III also do not affect the f-vector of $\mathscr{C}(V, 2m)/J$.

The main result of this section, Lemma 11.1, can be phrased as follows: Let D, D' be two (distinct) components of the graph $C(V) \setminus J$, each having at least $m - j + 2$ vertices. D and D' are paths, and the complement of $D \cup D'$ in $C(V)$ is also a union of two paths, call them A, B. The endpoints of A and B are in J. Let J' be the union of $J \cap B$ and the reflection of $J \cap A$ about the midpoint of A. Then $f(\mathscr{C}(V, 2m)/J) = f(\mathscr{C}(V, 2m)/J')$.

(Lemma 11.1, as it appears below, is phrased in different terms, in order to facilitate its inductive proof.)

The invariance of $f(\mathscr{C}(V, 2m)/J)$ under transformations of J of type II or III can be deduced from the above informal version of Lemma 11.1 as follows:

For type II, apply the lemma with D, D', $J \cap A$ replaced by D_i, D_{i+1} and B_i, respectively. (The sets B_i, D_i are defined in the beginning of Section 10.) For type III, apply the lemma three times: first with $D = D_i$, $D' = D_{i+2}$, then with $D = D_i$, $D' = D_{i+1}$, and finally with $D = D_{i+1}$, $D' = D_{i+2}$, as shown in the accompanying diagram. In the diagram, X^* denotes a reflection of the set X, and the dotted arcs indicate which segment is about to be reflected.



The notation used in the next lemma and its proof follows Definition 10.1 (see Fig. 3).

LEMMA 11.1. *Suppose* $1 \leqq \sigma < \tau \leqq j$, *and* $\min(l_\sigma, l_\tau) \geqq m - j + 2$. *Define*

$$l_i^* = \begin{cases} l_i & \text{if } i \leqq \sigma \quad \text{or} \quad \tau \leqq i, \\ l_{\sigma+\tau-i} & \text{if } \sigma < i < \tau. \end{cases}$$

Then $f(m; l_1, \cdots, l_j) = f(m; l_1^*, \cdots, l_j^*)$.

PROOF. Since the function $f(m; l_1, \cdots, l_j)$ is invariant under circular shifts and reflections of the sequence $l_1, \cdots, l_j$, we shall assume, without loss of generality, that $\sigma = 1$. The lemma holds trivially for $\tau = 2$ and $\tau = 3$, and for $\tau = j$ it follows from Theorem 10.3, and from the invariance properties of f mentioned above.

Assume therefore that $\sigma = 1$ and $4 \leqq \tau \leqq j - 1$. We also assume that $j < m$, since the case $j = m$ is already covered by Theorem 8.2.

Let $\mathcal{K} = \mathcal{K}(m; l_1, \cdots, l_j)$ and $\mathcal{K}^* = \mathcal{K}(m; l_1^*, \cdots, l_j^*)$. $\mathcal{K}$, $\mathcal{K}^*$ are $(2m - j - 1)$-complexes over $V \backslash J$. Fix k, $0 \leqq k < 2m - j$, and denote by $\mathcal{F}_k$, $\mathcal{F}_k^*$ the set of k-faces of $\mathcal{K}$, $\mathcal{K}^*$ respectively.

As in the proof of Theorem 10.4, let $h_1^+ = h_1 - 1$, $h_\tau^- = h_{\tau-1} + 1$ (see Fig. 3). Define subsets $\mathcal{F}_k(i)$, $0 \leqq i \leqq 3$, as follows:

$$\mathcal{F}_k(0) = \{F \in \mathcal{F}_k : h_1^+ \notin F \text{ and } h_\tau^- \notin F\}, \qquad \mathcal{F}_k(1) = \{F \in \mathcal{F}_k : h_1^+ \in F\},$$
$$\mathcal{F}_k(2) = \{F \in \mathcal{F}_k : h_\tau^- \in F\}, \qquad \mathcal{F}_k(3) = \mathcal{F}_k(1) \cap \mathcal{F}_k(2).$$

Then $\mathcal{F}_k = \mathcal{F}_k(0) \cup \mathcal{F}_k(1) \cup \mathcal{F}_k(2)$, and $\mathcal{F}_k(0) \cap (\mathcal{F}_k(1) \cup \mathcal{F}_k(2)) = \emptyset$. Therefore $|\mathcal{F}_k| = |\mathcal{F}_k(0)| + |\mathcal{F}_k(1)| + |\mathcal{F}_k(2)| - |\mathcal{F}_k(3)|$. In the same manner we define $\mathcal{F}_k^*(i)$ of $\mathcal{F}_k^*$ $(0 \leqq i \leqq 3)$. It suffices to show that $|\mathcal{F}_k(0)| = |\mathcal{F}_k^*(0)|$, $|\mathcal{F}_k(1)| = |\mathcal{F}_k^*(2)|$, $|\mathcal{F}_k(2)| = |\mathcal{F}_k^*(1)|$ and $|\mathcal{F}_k(3)| = |\mathcal{F}_k^*(3)|$.

A 1-1 correspondence between $\mathcal{F}_k(0)$ and $\mathcal{F}_k^*(0)$ is established as follows: For $S \subset [h_1, h_{\tau-1}]$, let $S' = \{h_1 + h_{\tau-1} - i : i \in S\}$. S' is the reflection of S about the midpoint of the interval $[h_1, h_{\tau-1}]$. For $F \subset V (= [1, v])$ define $F^* = (F \backslash [h_1, h_{\tau-1}]) \cup (F \cap [h_1, h_{\tau-1}])'$. Clearly $|F^*| = |F|$, and J^* is the set that corresponds to J in Definition 10.1 with respect to the sequence $(l_1^*, \cdots, l_j^*)$ (i.e., $\mathcal{K}^* = \mathcal{C}(V, 2m)/J^*$). If $F \subset V \backslash (J \cup \{h_1^+, h_\tau^-\})$, then $F^* \subset V \backslash (J^* \cup \{h_1^+, h_\tau^-\})$, and $\nu(J \cup F) = \nu(J^* \cup F^*)$. ($\nu(S)$ is the number of odd blocks of S.) Therefore $F \in \mathcal{F}_k(0)$ iff $F^* \in \mathcal{F}_k^*(0)$, hence $|\mathcal{F}_k(0)| = |\mathcal{F}_k^*(0)|$.

The equality $|\mathcal{F}_k(1)| = |\mathcal{F}_k^*(2)|$ will be proved by induction on the parameters k, j, m, as follows:

If $k = 0$, then $|\mathcal{F}_k(1)| = |\mathcal{F}_k^*(2)| = 1$. Assume therefore $k \geqq 1$. Then

$$\begin{aligned}|\mathcal{F}_k(1)| &= |\{F \subset V \backslash J : F \cup J \text{ is a face of } \mathcal{C}(V, 2m) \text{ and } |F| = k + 1 \text{ and } h_1^+ \in F\}| \\ &= |\{G \subset V \backslash (J \cup \{h_1^+\}) : G \cup J \cup \{h_1^+\} \text{ is a face of } \mathcal{C}(V, 2m) \text{ and } |G| = k\}| \\ &= f_{k-1}(\mathcal{C}(V, 2m)/(J \cup \{h_1^+\})).\end{aligned}$$

By [1, Theorem 3.5], this is equal to

$$(*)\qquad \begin{aligned} &f_{k-1}(\mathscr{C}(V\setminus\{h_1^+, h_1\}, 2(m-1))/(J\setminus\{h_1\})) \\ &\quad = f_{k-1}(m-1; l_1 + l_2 - 1, l_3, \cdots, l_{\tau-1}, l_\tau, \cdots, l_j). \end{aligned}$$

In a similar fashion we obtain

$$|\mathscr{F}_k^*(2)| = f_{k-1}(m-1; l_1, l_{\tau-1}, \cdots, l_3, l_2 + l_\tau - 1, l_{\tau+1}, \cdots, l_j).$$

By Theorem 10.3 with m, j replaced by $m-1$, $j-1$, this is equal to

$$(**)\qquad f_{k-1}(m-1, l_1 + l_2 - 1, l_{\tau-1}, \cdots, l_3, l_\tau, l_{\tau+1}, \cdots, l_j),$$

since $\min(l_1, l_\tau) \geqq (m-1) - (j-1) + 2$.

By the induction hypothesis of the present lemma, (*) and (**) are equal. This shows that $|\mathscr{F}_k(1)| = |\mathscr{F}_k^*(2)|$. The equality $|\mathscr{F}_k(2)| = |\mathscr{F}_k^*(1)|$ is proved in the same way.

Finally, we show by induction on k, j, m, that $|\mathscr{F}_k(3)| = |\mathscr{F}_k^*(3)|$. Recall that $\tau \geqq 4$, and note that there is nothing to prove unless $k \geqq 2$. As above, we obtain:

$$\begin{aligned} |\mathscr{F}_k(3)| &= f_{k-2}(\mathscr{C}(V\setminus\{h_1^+, h_1, h_{\tau-1}, h_\tau^-\}, 2(m-2))/(J\setminus\{h_1, h_{\tau-1}\})) \\ &= f_{k-2}(m-2; l_1 + l_2 - 1, l_3, \cdots, l_{\tau-2}, l_{\tau-1} + l_\tau - 1, l_{\tau+1}, \cdots, l_j), \end{aligned}$$

and similarly $|\mathscr{F}_k^*(3)| = f_{k-2}(m-2; l_1 + l_{\tau-1} - 1, l_{\tau-2}, \cdots, l_3, l_2 + l_\tau - 1, l_{\tau+1}, \cdots, l_j)$.

The expressions obtained for $|\mathscr{F}_k(3)|$ and $|\mathscr{F}_k^*(3)|$ are equal by Theorem 10.3 (with m, j replaced by $m-2$, $j-2$) and by the induction hypothesis of the present lemma.

This concludes the proof of Lemma 11.1, and with it the proof of the main result of this paper, Theorem 7.2.

12. Stability of the k-skeleton

Suppose $\mathscr{K} = \mathscr{C}(V, 2m)/J$, where $V = \{1, \cdots, v\}$, $v \geqq 2m+2$, $m \geqq 1$, J is a separated j-subset of V and $1 \leqq j \leqq m$. In part I we have seen that (for $j < m$ and $v \geqq 2m+3$) the isomorphism type of $\mathscr{K}$ determines the set J up to an isomorphism of the circuit $C(V)$. One may ask what changes of the set J do not affect the (isomorphism type of the) k-skeleton of $\mathscr{K}$, for a given value of k. In the sequel we shall give an answer to this question. The answer will have a certain formal resemblance to the corresponding result about the stability of the f-vector (Theorem 7.2).

Since $\mathscr{K}$ is $(m-j)$-neighborly (for $j < m$), $\mathrm{skel}_{m-j-1}\mathscr{K}$ is a complete

$(m-k-1)$-complex on $v-j$ vertices, and does not at all depend on J. Thus the above question is uninteresting for $k<m-j$. On the other hand, it is known (unpublished result of the second author) that the combinatorial type of a simplicial d-polytope is determined by the type of its $[d/2]$-skeleton. Therefore we cannot expect anything of interest if $k \geqq [\frac{1}{2}(2m-j)] = m - [\frac{1}{2}(j+1)]$. Thus the interesting range of k is $m-j \leqq k \leqq m - [\frac{1}{2}(j+3)]$.

If $S \subset V \setminus J$, $|S| \leqq k+1$, then $S \in \mathrm{skel}_k \mathcal{K}$ iff no subset of S is a missing face of $\mathcal{K}$ relative to $V \setminus J$ (see [1, Lemma 2.2 and Definition 4.4]). Therefore, if $\mathcal{K}' = \mathcal{C}(V,2m)/J'$, where J' is another separated j-subset of V, then a bijection $\varphi : V \setminus J \to V \setminus J'$ induces an isomorphism between $\mathrm{skel}_k \mathcal{K}$ and $\mathrm{skel}_k \mathcal{K}'$ iff for each subset M of $V \setminus J$ with $|M| \leqq k+1$, M is a missing face of $\mathcal{K}$ relative to $V \setminus J$ iff $\varphi(M)$ is a missing face of $\mathcal{K}'$ relative to $V \setminus J'$.

The set $V \setminus J$ splits into $t(=v-2j)$ chains $R_1, \cdots, R_t$ (see [1, Definition 4.2 and Lemma 4.3]), which we assume to be cyclically ordered in this order on the circuit $C(V)$. Assume $|R_i| = r_i$ for $1 \leqq i \leqq t$. Then $\Sigma_{i=1}^t r_i = v-j$, hence $j = \Sigma_{i=1}^t (r_i - 1)$, $v = \Sigma_{i=1}^t (2r_i - 1)$. Note that the subset J of V is determined (up to an automorphism of $C(V)$) by the sequence $r_1, \cdots, r_t$. The parameter m, however, is not determined by $r_1, \cdots, r_t$, and any value $j \leqq m \leqq \frac{1}{2}(v-2)$ will do. Similarly, denote by $R'_1, \cdots, R'_t$ and $r'_1, \cdots, r'_t$ the chains of $V \setminus J'$ and their respective lengths.

The above considerations justify the notation $\mathcal{K} = \mathcal{C}(V,2m)/J = \mathcal{K}(m; r_1, \cdots, r_t)$, $\mathcal{K}' = \mathcal{C}(V,2m)/J' = \mathcal{K}(m; r'_1, \cdots, r'_t)$.

The missing faces of $\mathcal{K}$ relative to $V \setminus J$ are precisely all the unions of $m-j+1$ separated chains of $V \setminus J$ (see [1, Definitions 4.2, 4.4 and Theorem 4.5]). Now if M is a union of $m-j+1$ chains, then each chain appearing in the union is of length $\leqq |M| - (m-j)$. Equality is possible only if all the chains in the union except one are singletons. Therefore, if $|M| \leqq k+1$, then the lengths of the chains included in M are bounded by α, where $\alpha = k+1-(m-j)$.

From the above discussion it follows that the following pair of conditions is sufficient for a bijection $\psi : V \setminus J \to V \setminus J'$ to induce an isomorphism between $\mathrm{skel}_k \mathcal{K}$ and $\mathrm{skel}_k \mathcal{K}'$:

(12.0.1) (a) If $S \subset V \setminus J$ and $|S| \leqq \alpha$, then S is a chain of $V \setminus J$ iff $\psi(S)$ is a chain of $V \setminus J'$.

(12.0.2) (b) If S_1, S_2 are chains of $V \setminus J$ of length $\leqq \alpha$, then S_1 and S_2 are adjacent (on $C(V)$) iff $\psi(S_1)$ and $\psi(S_2)$ are adjacent.

The statement of the main result of this section (Theorem 12.1) requires the following definitions.

Let $T=\{1,\cdots,t\}$. By $C(T)$ we denote, as usual, the undirected graph with vertex set T and edges $\{1,2\},\{2,3\},\cdots,\{t-1,t\},\{t,1\}$. For $S\subset T$, denote by $\langle S\rangle$ the subgraph of $C(T)$ spanned by S. We regard the sequences $R=\{r_i : i\in T\}$, $R'=\{r'_i : i\in T\}$ as systems of weights on the vertices of the graph $C(T)$. We always assume that r_i, r'_i are positive integers and that $\Sigma\{r_i : i\in T\}=\Sigma\{r'_i : I\in T\}$. For $\alpha\geqq 1$, define $T_\alpha(R)=\{i\in T : r_i\leqq\alpha\}$.

An *α-equivalence* between R and R' is a weight-preserving isomorphism between the weighted graphs $\langle T_\alpha(R)\rangle$ and $\langle T_\alpha(R')\rangle$. In other words, a bijection $\varphi : T_\alpha(R)\to T_\alpha(R')$ is an α-equivalence iff:

(12.0.3) (a) $r_i=r'_{\varphi(i)}$ for all $i\in T_\alpha(R)$,

(12.0.4) (b) if $i, j\in T_\alpha(R)$, then $i-j\equiv\pm1 \pmod t$ iff $\varphi(i)-\varphi(j)\equiv\pm1 \pmod t$.

A necessary and sufficient condition for α-equivalence between R and R' is the following: There is a (length-preserving) 1-1 correspondence $\langle C_i\rangle\to\langle C'_i\rangle$ between the components of $\langle T_\alpha(R)\rangle$ and those of $\langle T_\alpha(R')\rangle$, and for each $\langle C_i\rangle$, there is an automorphism ψ_i of $C(T)$ which maps C_i onto C'_i and is weight-preserving on C_i. (See example in Fig. 4, with $\alpha=2$ and $t=20$.)

From the above definition it follows that if R and R' are α-equivalent, then

(*) $$|\{i\in T : r_i>\alpha\}|=|\{i\in T : r'_i>\alpha\}| \quad\text{and}$$

(**) $$\Sigma\{r_i : i\in T, r_i>\alpha\}=\Sigma\{r'_i : i\in T, r'_i>\alpha\}.$$

THEOREM 12.1. *Suppose*

$$\mathcal{K}=\mathcal{C}(V,2m)/J=\mathcal{H}(m;r_1,\cdots,r_t),\quad \mathcal{K}'=\mathcal{C}(V,2m)/J'=\mathcal{H}(m;r'_1,\cdots,r'_t),$$

where $V=\{1,\cdots,v\}$, $v\geqq 2m+2$, J and J' are j-subsets of V, $j\leqq m$, and each of them is separated in $C(V)$. If the circular sequences $R=(r_1,\cdots,r_t)$ and $R'=(r'_1,\cdots,r'_t)$ are α-equivalent, then $\operatorname{skel}_k\mathcal{K}\approx\operatorname{skel}_k\mathcal{K}'$, *where $k=m-j-1+\alpha$.*

PROOF. Let $\varphi : T_\alpha(R)\to T_\alpha(R')$ be an α-equivalence between R and R'. For each $i\in T_\alpha(R)$, choose an arbitrary bijection $\psi_i : R_i\to R'_{\varphi(i)}$. Choose also a bijection $\psi_0 : \bigcup\{R_i : i\in T, |R_i|>\alpha\}\to\bigcup\{R'_i : i\in T, |R'_i|>\alpha\}$. (This is possible because of (**).) Finally, let $\psi=\psi_0\cup\bigcup\{\psi_i : i\in T_\alpha(R)\}$. Then ψ is a 1-1 correspondence between $V\setminus J$ and $V\setminus J'$, which satisfies conditions (12.0.1) and (12.0.2) (see (12.0.4)), and therefore induces an isomorphism between $\operatorname{skel}_k\mathcal{K}$ and $\operatorname{skel}_k\mathcal{K}'$. □

EXAMPLE. Figure 4 shows a pair of 2-equivalent cyclic sequences R, R' of

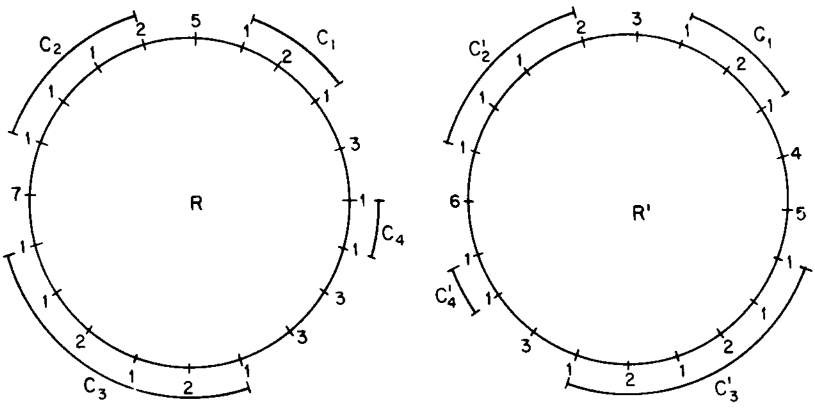


Fig. 4. 2-equivalence between R and R' with $t = 20$, four components and $\Sigma_{i=1}^{20} r_i = \Sigma_{i=1}^{20} r'_i = 40$.

length $t = 20$, with $\Sigma_{i=1}^{20} r_i = \Sigma_{i=1}^{20} r'_i = 40$. Each of the graphs $\langle T_2(R)\rangle$, $\langle T_2(R')\rangle$ has four components. These sequences correspond to quotients $\mathcal{K} = \mathcal{C}(V, 2m)/J$, $\mathcal{K}' = \mathcal{C}(V, 2m)/J'$, where $|V| = v = 60$, $|J| = |J'| = j = 20$, and m can take any value between 20 and 29. For each such choice of m, Theorem 12.1 implies that $\text{skel}_{m-19}\mathcal{K} \approx \text{skel}_{m-19}\mathcal{K}'$. However, a direct examination of $\mathcal{K}$ and $\mathcal{K}'$ for $m = 29$ reveals that $\text{skel}_k\mathcal{K} \approx \text{skel}_k\mathcal{K}'$ even for $k = 18$.

The proof of Theorem 12.1 took into consideration the worst possible case, namely that a missing face of $\mathcal{K}$ may consist of $m - j$ chains of length 1 and one long chain. Therefore if we assume that there are not too many separated short chains, then the conclusion of Theorem 12.1 can be strengthened as follows.

Fix α, $\alpha \geqq 1$. Call a subset U of $T_\alpha(R)$ *separated*, if no two points in U are adjacent in $C(T)$. Define, for $1 \leqq x \leqq m - j + 1$, $\mu(x, R) = \min\{\Sigma\{r_i : i \in U\} : U$ is a separated $(m - j + 1 - x)$-subset of $T_\alpha(R)\}$. (If there is no such set U, let $\mu(x, R) = \infty$.) Define also

$$k(R) = \min\{(\alpha + 1)x + \mu(x, R) : 1 \leqq x \leqq m - j + 1\} - 2.$$

Note that if R and R' $(= \{r'_i : i \in T\})$ are α-equivalent, then $\mu(x, R) = \mu(x, R')$ and $k(R) = k(R')$.

Let M be a missing face of $\mathcal{K}$ relative to $V \setminus J$. M consists of $m - j + 1$ separated chains. Suppose x of them are of length $> \alpha$, and the remaining $m - j + 1 - x$ are of length $\leqq \alpha$. Then $|M| \geqq (\alpha + 1)x + \mu(x, R)$. Therefore, if $x > 0$, then $|M| \geqq k(R) + 2$, and M does not affect the $k(R)$-skeleton of $\mathcal{K}$. Therefore, if R and R' are α-equivalent, and $\mathcal{K} = \mathcal{K}(m; r_1, \cdots, r_t)$, $\mathcal{K}' = \mathcal{K}(m; r'_1, \cdots, r'_t)$, then $\text{skel}_k\mathcal{K} \approx \text{skel}_k\mathcal{K}'$ for $k = k(R) = k(R')$.

If $r_i \geqq c$ for all $i \in T$, and $\alpha \geqq c-1$, then $\mu(x, R) \geqq (m-j+1-x)c$, and therefore $k(R) \geqq \alpha + 1 + (m-j)c - 2 = \alpha + (m-j)c - 1$. If $c = 1$, this reduces to $k(R) \geqq m - j + \alpha - 1$, as in Theorem 12.1.

For the example shown in Fig. 4, with $m = 29$, we obtain $\mu(1, R) = \infty$, $\mu(x, R) = 10 - x$ for $2 \leqq x \leqq 10$, thus $k(R) = 12$.

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